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Covariant time derivatives for dynamical systems

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Abstract

We present a unified derivation of covariant time derivatives, which transform as tensors under a time-dependent coordinate change. Such derivatives are essential for formulating physical laws in a frame-independent manner. Three specific derivatives are described: convective, corotational and directional. The covariance is made explicit by working in arbitrary time-dependent coordinates, instead of restricting to Eulerian or Lagrangian coordinates. The commutator of covariant time and space derivatives is interpreted in terms of a *time-curvature* that shares many properties of the Riemann curvature tensor, and reflects nontrivial time dependence of the metric.

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1. Introduction

In physics, choosing an appropriate coordinate system can make the difference between a tractable problem and one that defies analytical study. In fluid dynamics, two main types of coordinate are used, each representing a natural setting in which to study fluid motion: the Eulerian coordinates, also known as the laboratory frame, are time independent and fixed in space; in contrast, the Lagrangian (or material) coordinates are constructed to move with fluid elements. In between these extremes, other types of coordinate are used, such as rotating coordinates in geophysical fluid dynamics. Such coordinates usually have a nontrivial spatial and temporal dependence.

The situation becomes more complicated when dealing with moving surfaces: here the metric itself has intrinsic time dependence. This time dependence incorporates the strain imposed on a 2D surface flow as the surface deforms. To properly formulate fluid equations on thin films and other surfaces, one needs a *covariant* description, that is, a description of the building blocks of equations of motion—spatial and temporal derivatives—that obey tensorial transformation laws.

There are other reasons than inherent deformation of the space to introduce a time-dependent, nontrivial metric. For instance, the advection–diffusion equation can have an anisotropic, time-dependent diffusion tensor, perhaps arising from some inhomogeneous turbulent process. In that case, it is advantageous to use the diffusion tensor as a metric, for then the characteristic directions of stretching, given by the eigenvectors of the metric tensor in Lagrangian coordinates, correspond to directions of suppressed or enhanced diffusion associated with positive or negative Lyapunov exponents, respectively [1, 2].

Local physical quantities can be viewed as tensors (scalars, vectors, or higher-order tensors) evaluated along fluid trajectories. For instance, we may be interested in how the temperature (scalar) of a fluid element varies along a trajectory, or how the magnetic field (vector) associated with a fluid element evolves. Characterizing the evolution of these tensors in complicated coordinates is again best done using some form of covariant time derivative, also called an *objective* time derivative.

The covariant spatial derivative is a familiar tool of differential geometry [3–5]. The emphasis is usually on covariance under coordinate transformations of the full spacetime. In fluid dynamics and general dynamical systems, however, the time coordinate is not included in the metric (though the metric components may depend on time), and the required covariance is less restrictive: we seek covariance under time-dependent transformations of the coordinates, but the new time is the same as the old and does not depend on the coordinates. Time derivatives lead to non-tensorial terms because of time-dependent basis vectors—the same reason that ordinary derivatives are not covariant.

There are many ways of choosing a covariant time derivative. The most familiar is the *convective* derivative introduced by Oldroyd [6, 7] in formulating rheological equations of state. This derivative was then used by Scriven [8] to develop a theory of fluid motion on an interface. The convective derivative of a tensor is essentially its Lie derivative along the velocity vector. In spite of its economical elegance, the convective derivative has drawbacks. Firstly, unlike the usual covariant spatial derivative, it is not *compatible* with the metric tensor. A compatible operator vanishes when acting on the metric. Because the covariant derivative also has the Leibniz property, compatibility allows the raising and lowering of indices ‘through’ the operator. This is convenient for some applications [1], and implies that the equation of motion for a contravariant tensor has the same form as the covariant one. A second drawback of the convective derivative is that it involves gradients of the velocity, and so is not *directional*. The commutator of the convective derivative and the spatial derivative thus involves *second* derivatives of the velocity, requiring it to be at least of class C^2 .

A second common type of derivative is the *corotational* or *Jaumann* derivative (see [9] and [10, p 342], and references therein), where the local vorticity of the flow is incorporated into the derivative operator. The corotational derivative is compatible with the metric, but like the convective derivative it depends on gradients of the velocity.

The third type of derivative we discuss is a new, time-dependent version of the usual *directional* derivative along a curve used to define parallel transport [3–5]. The curve here is the actual trajectory of a fluid particle, with tangent vector given by the Eulerian velocity field. The directional derivative does not depend on gradients of the velocity field. The concept of time-dependent parallel transport can be introduced using this derivative, and is equivalent to a covariant description of advection without stretching. A directional derivative was introduced in the context of fluid motion by Truesdell [11, p 42], but it does not allow for time dependence in the coordinates or metric. (Truesdell calls it the *material* derivative because of its connexion to fluid elements.)

In this paper, we present a unified derivation of these different types of covariant time derivative. We do not restrict ourselves to Eulerian and Lagrangian coordinates, as this

obscures the general covariance of the theory: both these descriptions lack certain terms that vanish because of the special nature of the coordinates. From a dynamical system defined in some Eulerian frame, we transform to general time-dependent coordinates. We then find a transformation law between two time-dependent frames with no explicit reference to the Eulerian coordinates. The Eulerian velocity of the flow is not a tensor, but the move to general coordinates allows the identification of a *velocity tensor* that transforms appropriately (section 2). We also derive a time evolution equation for the Jacobian matrix of a coordinate transformation between two arbitrary time-independent frames. This time evolution equation facilitates the construction of the covariant time derivative in section 3. After a discussion of the rate-of-strain tensor in section 4, we present in section 5 the three types of covariant time derivative mentioned above: convective, corotational and directional.

Section 6 addresses a fundamental issue when dealing with generalized coordinates: the problem of commuting derivatives. In manipulating fluid equations it is often necessary to commute the order of time and space derivation. When commuting two covariant spatial derivatives, the Riemann curvature tensor must be taken into account. Similarly, when commuting a covariant time derivative with a spatial derivative, there arises a tensor we call the *time-curvature*. This tensor vanishes for sufficiently simple time dependence of the metric, and satisfies many properties similar to the Riemann tensor.

Throughout this paper, we will usually refer to the ‘fluid’, ‘fluid elements’ and ‘velocity’, but this is merely a useful concretion. The methods developed apply to general dynamical systems where the velocity is some arbitrary vector field defined on a manifold. The covariant time derivative still refers to the rate of change of tensors along the trajectory, but the tensors do not necessarily correspond to identifiable physical quantities. For example, the covariant time derivative is useful in formulating methods for finding Lyapunov exponents on manifolds with nontrivial metrics [1].

2. Time-dependent coordinates

We consider the dynamical system on an n -dimensional smooth manifold \mathcal{U} ,

$$\dot{x} = v(t, x) \quad (1)$$

where the overdot indicates a time derivative and v is a differentiable vector field. (For simplicity, we restrict ourselves to a given chart.) A solution $x(t)$ defines a curve \mathcal{C} in \mathcal{U} with tangent v . We view the x as special coordinates, called the Eulerian coordinates, and denote vectors expressed in the Eulerian coordinate basis $\{\partial/\partial x^i\}$ by the indices i, j, k .

A time-dependent coordinate change $z(t, x)$ satisfies

$$\dot{z}^a(t, x(t)) = \frac{\partial z^a}{\partial x^k} v^k + \frac{\partial z^a}{\partial t} \Big|_x \quad (2)$$

where the $\partial/\partial t|_x$ is taken at constant x . Here and throughout the rest of the paper, we assume the usual Einstein convention of summing over repeated indices. We denote vectors expressed in the general coordinate basis $\{\partial/\partial z^a\}$ by the indices a, b, c, d . We use the shorthand notation that the index on a vector X characterizes the components of that vector in the corresponding basis: thus X^a and X^i are the components of X in the bases $\{\partial/\partial z^a\}$ and $\{\partial/\partial x^i\}$, respectively. The components X^a and X^i are also understood to be functions of z and x , respectively, in addition to depending explicitly on time.

Defining $v := \dot{z}$, we can regard equation (2) as a transformation law for v ,

$$v^a = \frac{\partial z^a}{\partial x^k} v^k + \frac{\partial z^a}{\partial t} \Big|_x. \quad (3)$$

This last term prevents v from transforming like a tensor. (We refer the reader to standard texts in differential geometry for a more detailed discussion of tensors [3–5].)

Now consider a second coordinate system $\bar{z}(t, x)$, also defined in terms of x . We can use equation (3) and the chain rule to define a transformation law between z and \bar{z} ,

$$v^a - \frac{\partial z^a}{\partial t} \Big|_x = \frac{\partial z^a}{\partial \bar{z}^{\bar{a}}} \left(v^{\bar{a}} - \frac{\partial \bar{z}^{\bar{a}}}{\partial t} \Big|_x \right). \quad (4)$$

Any explicit reference to the coordinates x has disappeared (except in $\partial/\partial z|_x$). Equation (4) is a transformation law between any two coordinate systems defined in terms of x , and implies that $v^a - (\partial z^a/\partial t)|_x$ transforms like a tensor. This suggests defining the tensor

$$V^a := v^a - \frac{\partial z^a}{\partial t} \Big|_x \quad (5)$$

which we call the velocity tensor. The velocity tensor is the absolute velocity of the fluid v with the velocity of the coordinates subtracted.

In addition to the coordinates x , characterized by $\partial x^i/\partial t|_x = 0$, we introduce another special set of coordinates, the Lagrangian coordinates a , defined by $\dot{a} = 0$. We denote vectors expressed in the Lagrangian coordinate basis $\{\partial/\partial a^q\}$ by the indices p and q . From equation (3), we have

$$v^q(t, x(t)) = \frac{\partial a^q}{\partial x^k} v^k + \frac{\partial a^q}{\partial t} \Big|_x = 0. \quad (6)$$

The initial conditions for a are chosen such that Eulerian and Lagrangian coordinates coincide at $t = 0$: $a(0, x) = x$.

Lagrangian and Eulerian coordinates have the advantage that the time evolution of their Jacobian matrix is easily obtained. The Jacobian matrix $\partial x^i/\partial a^q$ satisfies [6]

$$\frac{d}{dt} \left(\frac{\partial x^i}{\partial a^q} \right) = \frac{\partial v^i}{\partial x^k} \frac{\partial x^k}{\partial a^q}. \quad (7)$$

By using the identity

$$\frac{d}{dt} \left(\frac{\partial x^i}{\partial a^q} \frac{\partial a^p}{\partial x^i} \right) = \frac{d}{dt} (\delta_q^p) = 0$$

which follows from the chain rule, and using the Leibniz property and equation (7), we find

$$\frac{d}{dt} \left(\frac{\partial a^q}{\partial x^i} \right) = - \frac{\partial a^q}{\partial x^k} \frac{\partial v^k}{\partial x^i}.$$

The Leibniz property can be used again to find the time evolution of the Jacobian matrix of two arbitrary time-dependent transformations $z(t, x)$ and $\bar{z}(t, x)$,

$$\frac{d}{dt} \left(\frac{\partial z^a}{\partial \bar{z}^{\bar{a}}} \right) = \frac{\partial v^a}{\partial z^b} \frac{\partial z^b}{\partial \bar{z}^{\bar{a}}} - \frac{\partial z^a}{\partial \bar{z}^{\bar{b}}} \frac{\partial v^{\bar{b}}}{\partial \bar{z}^{\bar{a}}}. \quad (8)$$

All reference to Eulerian and Lagrangian coordinates has disappeared from equation (8); this equation is crucial when deriving the covariant time derivative of section 3.

3. The covariant time derivative

The standard time derivative operator, which we have been denoting by an overdot, is defined for a vector field X as

$$\dot{X}^a := \frac{\partial X^a}{\partial t} \Big|_z + \frac{\partial X^a}{\partial z^b} v^b \quad (9)$$

where we recall that $\dot{z} = v$. The first term is the change in X due to any explicit time dependence it might have; the second term is the change in X due to its dependence on z . The time derivative is not covariant, because a time-dependent change of basis will modify the form of equation (9).

We define the covariant time derivative \mathcal{D} by

$$\mathcal{D}X^a := \dot{X}^a + \alpha^a_b X^b \quad (10)$$

where the α^a_b are time-dependent quantities that are chosen to make $\mathcal{D}X^a$ covariant. In order that the operator \mathcal{D} have the Leibniz property, and that it reduce to the ordinary derivative (9) when acting on scalars, we require

$$\mathcal{D}Y_a = \dot{Y}_a - \alpha^b_a Y_b$$

when acting on a 1-form Y . When \mathcal{D} acts on mixed tensors of higher rank, an α must be added for each superscript, and one must be subtracted for each subscript. We refer to the α as *connexions*, by analogy with the spatial derivative case.

By enforcing covariance of \mathcal{D} , we can derive a general expression for α^a_b . Since X is a tensor, we can write

$$\begin{aligned} \mathcal{D}X^a &= \mathcal{D}\left(\frac{\partial z^a}{\partial \bar{z}^{\bar{a}}} X^{\bar{a}}\right) \\ &= \frac{d}{dt}\left(\frac{\partial z^a}{\partial \bar{z}^{\bar{a}}}\right) X^{\bar{a}} + \frac{\partial z^a}{\partial \bar{z}^{\bar{a}}} \dot{X}^{\bar{a}} + \frac{\partial z^b}{\partial \bar{z}^{\bar{a}}} X^{\bar{a}} \alpha^a_b \\ &= \frac{\partial z^a}{\partial \bar{z}^{\bar{a}}} \mathcal{D}X^{\bar{a}} \end{aligned}$$

because $\mathcal{D}X^a$ is by definition covariant. Hence, we require the α to transform as

$$\alpha^{\bar{a}}_{\bar{b}} = \frac{\partial \bar{z}^{\bar{a}}}{\partial z^a} \frac{\partial z^b}{\partial \bar{z}^{\bar{b}}} \alpha^a_b + \frac{\partial \bar{z}^{\bar{a}}}{\partial z^c} \frac{d}{dt} \left(\frac{\partial z^c}{\partial \bar{z}^{\bar{b}}} \right). \quad (11)$$

The first term in (11) is the usual tensorial transformation law. The second term implies that α is not a tensor, and arises because of the time dependence.

Inserting the evolution equation (8) into (11), we can rewrite the transformation law for α as

$$\alpha^{\bar{a}}_{\bar{b}} + \frac{\partial v^{\bar{a}}}{\partial \bar{z}^{\bar{b}}} = \frac{\partial \bar{z}^{\bar{a}}}{\partial z^a} \frac{\partial z^b}{\partial \bar{z}^{\bar{b}}} \left(\alpha^a_b + \frac{\partial v^a}{\partial z^b} \right) \quad (12)$$

implying that $\alpha^a_b + (\partial v^a / \partial z^b)$ transforms like a tensor. Hence,

$$\alpha^a_b = -\frac{\partial v^a}{\partial z^b} + \mathcal{H}^a_b \quad (13)$$

where \mathcal{H} is an arbitrary tensor. Equation (13) is the most general form of the connexions α .

In section 5, we consider three convenient choices of the tensor \mathcal{H} . But first in section 4 we examine the action of the covariant derivative on the metric tensor.

4. The rate-of-strain tensor

Our development so far has not made use of a metric tensor. We now introduce such a tensor, specifically a Riemannian metric $g : \mathcal{TU} \times \mathcal{TU} \rightarrow \mathbb{R}$. The components g_{ab} of the metric are functions of z and t , but the indices a and b run over the dimension n of \mathcal{TU} , and so do not include a time component.

It is informative to consider the derivative of the metric tensor,

$$\begin{aligned}\mathcal{D}g_{ab} &= \dot{g}_{ab} - \alpha^c{}_a g_{bc} - \alpha^c{}_b g_{ac} \\ &= \frac{\partial g_{ab}}{\partial t} \Big|_z + \frac{\partial g_{ab}}{\partial z^c} v^c + g_{ac} \frac{\partial v^c}{\partial z^b} + g_{bc} \frac{\partial v^c}{\partial z^a} - (\mathcal{H}_{ab} + \mathcal{H}_{ba})\end{aligned}$$

where we have used the metric to lower the indices on \mathcal{H} . We define the intrinsic *rate-of-strain* or *rate-of-deformation* tensor γ [7, 8] as

$$\gamma_{ab} = \frac{1}{2} \left[g_{ac} \nabla_b v^c + g_{bc} \nabla_a v^c + \frac{\partial g_{ab}}{\partial t} \Big|_z \right]. \quad (14)$$

Here we denote by ∇_a the covariant derivative with respect to z^a ,

$$\nabla_b X^a := \frac{\partial X^a}{\partial z^b} + \Gamma_{bc}^a X^c.$$

The Riemann–Christoffel connexions are defined as [4, 5]

$$\Gamma_{bc}^a := \frac{1}{2} g^{ad} \left(\frac{\partial g_{bd}}{\partial z^c} + \frac{\partial g_{cd}}{\partial z^b} - \frac{\partial g_{bc}}{\partial z^d} \right) \quad (15)$$

whence the identity

$$g_{ac} \Gamma_{bd}^c + g_{bc} \Gamma_{ad}^c = \frac{\partial g_{ab}}{\partial z^d}$$

holds. The covariant time derivative of the metric can thus be rewritten

$$\frac{1}{2} \mathcal{D}g_{ab} = \gamma_{ab} - \mathcal{H}_{ab}^S \quad (16)$$

where $\mathcal{H}^S := \frac{1}{2}(\mathcal{H}_{ab} + \mathcal{H}_{ba})$ denotes the symmetric part of \mathcal{H} .

The rate-of-strain tensor γ describes the stretching of fluid elements. The time derivative of the metric in its definition (14) is necessary for covariance under time-dependent transformations; the term describes straining motion that is inherent to the space, as embodied by the metric. The trace of the rate-of-strain tensor is a scalar

$$\gamma^c{}_c = g^{ac} \gamma_{ac} = \nabla_c v^c + \frac{1}{2} \frac{\partial}{\partial t} \Big|_z \log |g| \quad (17)$$

where $|g|$ is the determinant of g_{ab} and we have used the identity

$$g^{ac} \frac{\partial g_{ac}}{\partial t} \Big|_z = \frac{\partial}{\partial t} \Big|_z \log |g|. \quad (18)$$

The rate-of-strain tensor can be decomposed as

$$\gamma_{ab} = \gamma'_{ab} + \frac{1}{n} \gamma^c{}_c g_{ab} \quad (19)$$

where γ'_{ab} is traceless and represents a straining motion without change of volume, and $\gamma^c{}_c g_{ab}/n$ is an isotropic expansion. We see from the trace (17) that for a time-dependent metric there can be an isotropic expansion even for an incompressible flow, if $\partial|g|/\partial t|_z \neq 0$. Note also that in Lagrangian coordinates (characterized by $v^q = 0$), the rate-of-strain tensor reduces to

$$\gamma_{pq} = \frac{1}{2} \frac{\partial g_{pq}}{\partial t} \Big|_a$$

so that the deformation of the space is contained entirely in the metric tensor.

Table 1. Comparison of the equation of motion for the components of an advected and stretched vector field B . The equations for the covariant and contravariant components of $\mathcal{D}_c B$ differ because of the lack of compatibility with the metric.

Type	Contravariant components	Covariant components
Convective	$\mathcal{D}_c B^a = 0$	$\mathcal{D}_c B_a = 2B_c \gamma^c_a$
Corotational	$\mathcal{D}_J B^a = B^c \gamma^a_c$	$\mathcal{D}_J B_a = B_c \gamma^c_a$
Directional	$\mathcal{D}_v B^a = B^c (\nabla_c V^a + \kappa_c^a)$	$\mathcal{D}_v B_a = B_c (\nabla^c V_a + \kappa_a^c)$

5. Three covariant derivatives

As mentioned in section 2, the requirement of covariance only fixes the covariant time derivative up to an arbitrary tensor (equation (13)). That tensor may be chosen to suit the problem at hand, but there are three particular choices that merit special attention. In section 5.1 we treat the convective derivative, and in section 5.2 we examine two types of compatible derivative: corotational and directional.

5.1. The convective derivative

The choice $\mathcal{H} \equiv 0$ is equivalent to the convective derivative of Oldroyd [6, 7]. The connexion, equation (13), reduces to the simple form

$$\alpha^a_b = -\frac{\partial v^a}{\partial z^b}.$$

The convective derivative \mathcal{D}_c acting on a vector X^a is thus

$$\mathcal{D}_c X^a = \left. \frac{\partial X^a}{\partial t} \right|_z + \frac{\partial X^a}{\partial z^b} v^b - \frac{\partial v^a}{\partial z^b} X^b. \quad (20)$$

When acting on a contravariant vector X^a , as in equation (20), \mathcal{D}_c is sometimes called the *upper convected* derivative [9]; \mathcal{D}_c acting on a covariant vector Y_a , $\mathcal{D}_c Y_a = \dot{Y}_a + (\partial v^b / \partial z^a) Y_b$, is then called the *lower convected* derivative.

In general, for an arbitrary tensor \mathcal{T} ,

$$\mathcal{D}_c \mathcal{T} = \left. \frac{\partial \mathcal{T}}{\partial t} \right|_z + \mathcal{L}_v \mathcal{T}$$

where $\mathcal{L}_v \mathcal{T}$ is the Lie derivative of \mathcal{T} with respect to v [4, 5]. In Lagrangian coordinates, we have $v^a \equiv 0$, so the convective derivative reduces to

$$\mathcal{D}_c X^a = \left. \frac{\partial X^a}{\partial t} \right|_a.$$

The convective derivative is not compatible with the metric: from equation (16), the metric's derivative is

$$\frac{1}{2} \mathcal{D}_c g_{ab} = \gamma_{ab}$$

which does not vanish, unless the velocity field is strain-free.

The convective derivative is ideally suited to problems of *advection* with stretching, where a tensor is carried and stretched by a velocity field. Table 1 summarizes the form of the equation for advection with stretching of a vector field B (B is 'frozen in' the flow [12]) for the three different types of derivative introduced here. The equation for the contravariant component B^a is simply $\mathcal{D}_c B^a = 0$, but the equation for the covariant component $B_a = g_{ac} B^c$ is $\mathcal{D}_c B_a = 2B_c \gamma^c_a$. These two equations differ because the operator \mathcal{D}_c is not compatible with the metric.

5.2. Compatible derivatives

Another way to fix \mathcal{H} is to require that the operator \mathcal{D} be compatible with the metric, that is, $\mathcal{D}g_{ab} = 0$. This allows us to raise and lower indices through the operator \mathcal{D} , a property possessed by the covariant spatial derivative. From equation (16), the requirement $\mathcal{D}g_{ab} = 0$ uniquely specifies the symmetric part of \mathcal{H}_{ab} , so that $\mathcal{H}^S = \gamma$. Using equations (13) and (14), we then find

$$\alpha_{ab} = g_{ac} \Gamma_{bd}^c v^d + \frac{1}{2} \frac{\partial g_{ab}}{\partial t} \Big|_z - \frac{1}{2} [g_{ac} \nabla_b v^c - g_{bc} \nabla_a v^c] + \mathcal{H}_{ab}^A \quad (21)$$

where $\mathcal{H}^A := \frac{1}{2}(\mathcal{H}_{ab} - \mathcal{H}_{ba})$ is the antisymmetric part of \mathcal{H} .

We define the antisymmetric vorticity tensor

$$\omega_{ab} := \frac{1}{2} \left[g_{ac} \frac{\partial V^c}{\partial z^b} - g_{bc} \frac{\partial V^c}{\partial z^a} \right] \quad (22)$$

and the symmetric coordinate rate-of-strain tensor

$$\kappa_{ab} := \frac{1}{2} \left[g_{ac} \nabla_b \left(\frac{\partial z^c}{\partial t} \Big|_x \right) + g_{bc} \nabla_a \left(\frac{\partial z^c}{\partial t} \Big|_x \right) + \frac{\partial g_{ab}}{\partial t} \Big|_z \right]. \quad (23)$$

In Eulerian coordinates, we have $\kappa_{ij} = \frac{1}{2}(\partial g_{ij}/\partial t)|_x$. The compatible connexion (21) can be rewritten

$$\alpha_{ab} = g_{ac} \Gamma_{bd}^c v^d - g_{ac} \nabla_b \left(\frac{\partial z^c}{\partial t} \Big|_x \right) + \kappa_{ab} - \omega_{ab} + \mathcal{H}_{ab}^A. \quad (24)$$

Since \mathcal{H}_{ab}^A is antisymmetric, we can use it to cancel the vorticity, or we can set it to zero. The two choices are discussed separately in sections 5.2.1 and 5.2.2.

The decomposition of the velocity gradient tensor ∇V into the rate-of-strain and vorticity tensors has the form

$$g_{ac} \nabla_b V^c = [\gamma_{ab} - \kappa_{ab}] + \omega_{ab} \quad (25)$$

in general time-dependent coordinates. When the coordinates have no time dependence, the tensor κ vanishes, as does the derivative $\partial z^c/\partial t|_x$, and we recover the usual decomposition of the velocity gradient tensor into the rate-of-strain and the vorticity. We can think of κ as the contribution to the rate-of-strain tensor that is due to coordinate deformation and not to gradients of the velocity field. However, the term $\partial g/\partial t|_z$ is a ‘real’ effect representing the deformation due to a time-dependent metric, and is thus also included in the definition of the intrinsic rate-of-strain tensor, γ , defined by equation (14).

In Euclidean space, when the rate-of-strain tensor γ vanishes everywhere we are left with rigid-body rotation at a constant rate given by ω [7]. With an arbitrary metric and time-dependent coordinates the situation is not so simple: the very concept of rigid-body rotation is not well defined. Hence, even when $\gamma \equiv 0$, we cannot expect to be able to solve for v in closed form.

5.2.1. The corotational derivative. In this instance we choose the antisymmetric part \mathcal{H}_{ab}^A to be zero. We call the resulting covariant derivative *corotational*, and denote it by \mathcal{D}_J (the subscript J stands for *Jaumann*). The appellation ‘corotational’ really applies to the Euclidean limit, $g_{ij} = \delta_{ij}$, for which the compatible connexion equation (21) reduces to $\alpha_{ij} = -\omega_{ij}$. It is then clear that the covariant derivative is designed to include the effects of local rotation of the flow, as embodied by the vorticity (see [9] and [10, p 342], and references therein). The derivative (21) with $\mathcal{H}^A \equiv 0$ is thus a generalization of the corotational derivative to include the effect of time-dependent non-Euclidean coordinates.

In table 1, we can see that, written using \mathcal{D}_J , the equation for advection with stretching of a vector B^a has the rate-of-strain tensor on the right-hand side. The ‘rotational’ effects are included in \mathcal{D}_J , hence the terms that remain include only the strain.

5.2.2. The directional derivative. Another convenient choice is to set $\mathcal{H}_{ab}^A = \omega_{ab}$, thus cancelling the vorticity in equation (24). The resulting covariant time derivative then has the property that, in the absence of any explicit time dependence, it reduces to the covariant derivative along the curve \mathcal{C} [4, 5], or directional derivative, where \mathcal{C} is the trajectory of the dynamical system in the general coordinates z (section 2). The derivative is called directional because it only depends on v , and not gradients of v .

The form of the equation for advection with stretching of a vector B^a written using \mathcal{D}_v is shown in table 1. The ∇V term on the right-hand side is the ‘stretching’ term [12] (called *vortex stretching* when B is the vorticity vector [13]). The κ term represents coordinate stretching, and does not appear in Euclidean space with time-independent coordinates.

Because the directional derivative depends only on v and not its gradients, it can be used to define time-dependent parallel transport of tensors. A vector X is said to be *parallel transported* along v if it satisfies $\mathcal{D}_v X = 0$, or equivalently

$$\left. \frac{\partial X^a}{\partial t} \right|_z + v^c \nabla_c X^a = X^c \left[\nabla_c \left(\left. \frac{\partial z^a}{\partial t} \right|_x \right) - \kappa^a{}_c \right]. \quad (26)$$

This can be readily generalized to tensors of higher rank. In Euclidean space, with time-independent coordinates, the right-hand side of equation (26) vanishes, leaving only advection of the components of X . Thus, parallel transport is closely related to advection without stretching; equation (26) is the covariant formulation of the passive advection equation.

6. Time-curvature

A hallmark of generalized coordinates is the possibility of having nonzero curvature. The curvature reflects the lack of commutativity of covariant derivatives, and is tied to parallel transport of vectors along curves [4, 5]. An analogous curvature arises when we try to commute \mathcal{D} and ∇ , respectively the covariant time and space derivatives:

$$\nabla_a [\mathcal{D} X^b] - \mathcal{D} [\nabla_a X^b] = \mathcal{H}^c{}_a \nabla_c X^b + g^{bc} [\nabla_a (\mathcal{H}_{cd} - \gamma_{cd} - \omega_{cd}) + R_{cdae} V^e + \frac{1}{2} S_{cda}] X^d \quad (27)$$

where the time-curvature tensor is defined by

$$S_{abc} := \nabla_a \left[\left. \frac{\partial g_{cb}}{\partial t} \right|_z + g_{be} \nabla_c \left(\left. \frac{\partial z^e}{\partial t} \right|_x \right) \right] - \nabla_b \left[\left. \frac{\partial g_{ca}}{\partial t} \right|_z + g_{ae} \nabla_c \left(\left. \frac{\partial z^e}{\partial t} \right|_x \right) \right] + R_{abce} \left. \frac{\partial z^e}{\partial t} \right|_x \quad (28)$$

and the Riemann curvature tensor R obeys [5]

$$(\nabla_c \nabla_d - \nabla_d \nabla_c) X^a = R^a{}_{bcd} X^b. \quad (29)$$

The time-curvature tensor satisfies $S_{abc} = -S_{bac}$, and the Riemann curvature tensor satisfies $R_{abcd} = -R_{bacd}$, $R_{abcd} = R_{cdab}$.

Even for trivial (Euclidean) coordinates, we do not expect \mathcal{D} and ∇ to commute, because of the derivatives of v in the $\nabla_a (\mathcal{H}_{cd} - \gamma_{cd} - \omega_{cd})$ term of equation (27). Note that the coordinate rate-of-strain tensor κ , defined by equation (23), does not appear in equation (27).

The ∇X term in equation (27) vanishes for the convective derivative of section 5.1, since then $\mathcal{H} \equiv 0$. For the directional derivative of section 5.2.2, we have $\mathcal{H}_{cd} = \gamma_{cd} + \omega_{cd}$, so the commutation relation simplifies to

$$\nabla_a[\mathcal{D}X^b] - \mathcal{D}[\nabla_a X^b] = (\gamma_{cd} + \omega_{cd}) \nabla_c X^b + g^{bc} [R_{cdae} V^e + \frac{1}{2} S_{cda}] X^d$$

which does not involve second derivatives of v . For the corotational derivative of section 5.2.1, with $\mathcal{H}_{cd} = \gamma_{cd}$, no terms drop out.

The terms involving \mathcal{H} in the commutation relation (27) reflect properties of the velocity field v . In contrast, the tensors R and S embody intrinsic properties of the metric tensor g . The Riemann tensor R is nonzero when the space is curved. The time-curvature tensor S is new and has characteristics that are analogous to the Riemann tensor. It satisfies a cyclic permutation identity,

$$S_{abc} + S_{cab} + S_{bca} = 0 \quad (30)$$

which corresponds to the first Bianchi identity of the Riemann tensor. The time-curvature does not appear to satisfy an analogue of the second Bianchi identity.

The property $S_{abc} = -S_{bac}$, together with the Bianchi identity (30), imply that S has $n(n^2 - 1)/3$ independent components, compared with the $n^2(n^2 - 1)/12$ components of R , where n is the dimension of the space. Thus one-dimensional manifolds have vanishing S and R . For $1 \leq n \leq 3$, S has more independent components than R ; for $n = 4$, they both have 20. For $n > 4$, R has more independent components than S .

The time-curvature S vanishes for a time-independent metric and coordinates. It also vanishes for a metric of the form $g_{ij}(t, x) = \beta(t) h_{ij}(x)$, where h is a time-independent metric and x are the Eulerian coordinates. It follows from its tensorial nature that the time-curvature must then vanish in any time-dependent coordinates. In general, it is convenient to find S in Eulerian coordinates (where $\partial x / \partial t|_x = 0$),

$$S_{ijk} := \nabla_i \left(\frac{\partial g_{kj}}{\partial t} \Big|_z \right) - \nabla_j \left(\frac{\partial g_{ki}}{\partial t} \Big|_z \right) \quad (31)$$

and then transform S_{ijk} to arbitrary time-dependent coordinates using the tensorial law.

7. Discussion

In this paper, we aimed to provide a systematic framework to handle complicated time-dependent metrics and coordinate systems on manifolds. The explicit form of the relevant tensors is often fairly involved, but the advantage is that they can be evaluated in time-independent Eulerian coordinates and then transformed to arbitrary coordinate systems using the usual tensorial transformation laws.

The covariance of the time derivatives is made explicit by using arbitrary time-dependent coordinates. The results for the Eulerian coordinates x^i are recovered by setting $\partial x^i / \partial t|_x = 0$, and those for the Lagrangian coordinates a^q by setting $v^q = 0$.

The introduction of the time-curvature tensor allows us to treat the temporal dependence of the metric tensor in a manner analogous to its spatial dependence. For simple time dependence, the time-curvature vanishes, such as for the case of a time-independent metric multiplied by a time-dependent scalar. As for the (spatial) Riemann curvature tensor, the components of the time-curvature can be computed for a given metric, and then inserted whenever a temporal and spatial derivative need to be commuted.

We have only addressed the *kinematics* of fluid motion. The dynamical equations relating the rate of change of quantities to the forces in play have not been discussed (see [6–9]), and depend on the specifics of the problem at hand. Nevertheless, covariant time derivatives provide a powerful framework in which to formulate such dynamical equations.

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